Entropic uncertainty relations for a quantum oscillator

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Abstract. We calculated the Shannon entropy of position and momentum for the stationary quantum states of the harmonic oscillator as a function of its energy and determined the corresponding entropic uncertainty relations for them. We found an approximate phenomenological function for the dependence of position and momentum entropies on the large quantum numbers and the corresponding asymptotic entropy–energy relation for the stationary harmonic oscillator. We also studied the time evolution of the position and momentum entropies of the non-stationary harmonic oscillator for the coherent states, squeezed vacuum and Schrödinger cat states.

1. Introduction

There has been considerable recent interest in the information-theoretic (or entropic) uncertainty relations (EUR) which represents an alternative formulation of the Heisenberg uncertainty principle (UP). In the information-theoretical formulation of UP, for expressing the uncertainty of an observable one uses the Shannon entropy of its probability distribution instead of its variance (see e.g. [1, 2, 20] and references therein). EUR represents the sum of entropies of two observables and is constructed as follows: consider a normalized state vector $|\psi\rangle$ in an N-dimensional Hilbert space and let the observables A and B have non-degenerate spectra of eigenvectors $|a_i\rangle$ and $|b_j\rangle$, respectively. The entropic uncertainty relation is an inequality of the form

$$S^{(A)} + S^{(B)} \geqslant S_{AB} \tag{1}$$

where

$$S^{(A)} = -\sum_i |\langle \psi | a_i \rangle|^2 \ln |\langle \psi | a_i \rangle|^2$$
$$S^{(B)} = -\sum_j |\langle \psi | b_j \rangle|^2 \ln |\langle \psi | b_j \rangle|^2 \tag{2}$$

and S_{AB} is a positive constant which represents the lower bound of the right-hand side of the inequality (1). For the continuous observables X_c and P_c described by the wavefunctions $\psi(x)$ and $\varphi(p)$, the relation (1) reads

$$S^{(X_c} + S^{(P_c)} \geqslant S_{XP} \tag{3}$$

where

$$S^{(X_c)} = -\int_{-\infty}^{\infty} |\psi(q)|^2 \ln |\psi(q)|^2 \, dq$$
$$S^{(P_c)} = -\int_{-\infty}^{\infty} |\varphi(p)|^2 \ln |\varphi(p)|^2 \, dp \tag{4}$$

represent the differential entropies of X_c and P_c. Białynicki-Birula and Mycielski [3], Deutsch [4], Maassen and Uffink [5], Sánchez-Ruiz [6] and Białynicki-Birula [7] have shown that non-trivial lower bounds for S_{AB} or S_{XP} exist for any two observables with no common eigenstates and found their values. The majority of authors dealing with the EUR have been trying to find the lower bound of the entropic UR, so, for example, Białynicki-Birula and Mycielski [3] found that for the sum of entropies of position and momentum it holds that

$$S^{(x)} + S^{(p)} \geqslant 1 + \ln \pi \tag{5}$$

(here $\hbar = 1$). Recently, there has been considerable interest in finding the dependence of $S^{(x)}$, $S^{(p)}$ and $S_{(xp)}$ on the quantum states for which $S_{(xp)}$ has larger values than its minimal one [16–19]. The dependence of $S^{(x)}$ and $S^{(p)}$ on the quantum state of the quantum harmonic oscillator (QHO) is interesting for the following reasons:

(i) Since the entropy is a measure of the spatial 'spreading out' of the wavefunction, its value for the various quantum states gives the picture of how the localization of the mass point of a QHO depends on these states.

(ii) Due to the Jaynes relation between the information-theoretical and physical entropy via Boltzmann's constant, k_B, one can ascribe to any quantum object a certain value of its physical entropy $S_{\text{phys}} = k_B S_{\text{inf}}$ [8].

(iii) The entropy–energy relation for QHO also plays an important role in the determination of the fundamental physical limit of computation [9].

This paper addresses the question of what are the values $S^{(x)}$, $S^{(p)}$ and $S^{(xp)}$ in EUR for the stationary and certain non-stationary states of a harmonic oscillator if x and p are its coordinate and momentum. Apart from being of interest in its own right, there is a specific motivation for studying these issues. The principal one concerns the question of the emergence of the classical behaviour of a quantum oscillator in the dependence of its state and the question of time dependence of EUR for some non-stationary states. In what follows we calculate the Shannon entropy for the position and momentum for the states of the QHO. We then consider the classical harmonic oscillator and calculate its position entropy in dependence of its total energy. We compare both the position entropies and find the asymptotic analytic formula for the dependence of the position entropy of a quantum oscillator on its energy. We show that the values of the position entropy of the QHO approach those of the corresponding entropy of the classical oscillator (CHO) as the quantum number n increases. This points out that for high values of n QHO can be treated classically, and also its position entropy. Finally, we calculate the time evolution of the position and momentum entropies for some non-stationary states of QHO and show that the squeezed vacuum states can get arbitrary values either for the position or momentum entropy, but their sum never drops below the entropy of the vacuum state as stated by the EUR (5). The Schrödinger cat states entropy oscillates between a maximum value when the Gaussian peaks are separated and a minimum value when the peaks interfere; for high excitations the difference between the minimum and the maximum entropies is just one *nat* (natural unit of information).

Let us recall the exact definition of the Shannon entropy as a *measure of the probability uncertainty* of a stochastic variable $\tilde{x}$. Let $\tilde{x}$ be a random variable defined on a stochastic object with the probability distribution function $\boldsymbol{P}$. The information-theoretical (or Shannon) entropy of the *random variable $\tilde{x}$, S,* is given by the general integral [10]

$$S(\tilde{x}) = -\int \boldsymbol{P}(\mathrm{d}x) \log \boldsymbol{P}(\mathrm{d}x). \tag{6}$$

For a continuous random variable $\tilde{x}_c$, the probability density function, $p(x)$, is given as

$$p(x) = \mathrm{d}\boldsymbol{P}/\mathrm{d}x \tag{7}$$

and the Shannon entropy has the form

$$S(\tilde{x}_c) = -\int p(x) \ln p(x) \, dx . \tag{8}$$

2. Entropy of the stationary states of QHO

2.1. EUR for a quantum harmonic oscillator

In this section we calculate $S^{(x)}$ and $S^{(p)}$ for the stationary states and compare the obtained values to the lower bound (5) and also to the results given by the classical theory of harmonic oscillators. In order to determine S^x and S^p for the quantum harmonic oscillator we need its wavefunctions in a coordinate and momentum representation which are the solutions of the corresponding Schrödinger equations for the harmonic oscillator. With the dimensionless abbreviation $\epsilon = E/(\hbar\omega)$ and $\xi = x/\sigma_x$, where $\sigma_x = \sqrt{\hbar/(m\omega)}$, one obtains the Schrödinger equation for the QHO in the coordinate representation in the form [11]

$$-\frac{d^2\Psi}{d\xi^2} + \xi^2\Psi = 2\epsilon\Psi . \tag{9}$$

This equation has an n-solution $\Psi_n = C_n H_n(\xi) \exp(-\xi^2/2)$ with the energies $\epsilon_n = n + \frac{1}{2}$, ($n = 0, 1, 2, \ldots$). Here $C_n = (\sqrt{\pi} 2^n n!)^{-1/2}$ and $H_n(\xi)$ are the Hermite polynomials of nth degree. The probability density for finding the mass point of QHO in the coordinate ξ is

$$\rho_n(\xi) = |\Psi_n|^2 = C_n^2 H_n^2(\xi) \exp(-\xi^2) \tag{10}$$

or, in terms of the x coordinate

$$\rho_n(x) = \frac{C_n^2}{\sigma_x} H_n^2(x/\sigma_x) \exp(-x^2/\sigma_x^2) . \tag{11}$$

Its position entropy is according to (8) given as

$$S_q^{(x)}(n) = -C_n^2 \int_{-\infty}^{\infty} H_n^2(\xi) \exp(-\xi^2) \ln\left[\frac{C_n^2}{\sigma_x} H_n^2(\xi) \exp(-\xi^2)\right] d\xi$$

$$= -C_n^2 \int_{-\infty}^{\infty} H_n^2(\xi) \exp(-\xi^2) \ln\left[C_n^2 H_n^2(\xi) \exp(-\xi^2)\right] d\xi + \ln\sigma_x . \tag{12}$$

Here the subscript q refers to the entropy of the QHO, in contrast to the subscript c with which we will next denote the quantities referring to the CHO. Since the integrals (12) cannot be generally evaluated analytically we determine them numerically, taking for the integration boundaries the multiple of the area where the position probability density function is most concentrated.

Since the wavefunction in the momentum representation is the Fourier transform of coordinate wavefunction, which can be expressed in a similar form to be coordinate wavefunction, we get for the momentum entropy the expression

$$S_q^{(p)}(n) = -C_n^2 \int_{-\infty}^{\infty} H_n^2(\xi) \exp(-\xi^2) \ln\left[C_n^2 H_n^2(\xi) \exp(-\xi^2)\right] d\xi + \ln\sigma_p \tag{13}$$

where $\sigma_p = \sqrt{\omega\hbar m}$. The values of the integrals can also be determined numerically and are equal to those of position entropies (up to a constant $\ln(\sigma_p/\sigma_x)$). Summing both entropies together we get EUR for the quantum states of QHO as a function of quantum number n

$$S_q^{(x)}(n) + S_q^{(p)}(n) = -2C_n^2 \int_{-\infty}^{\infty} H_n^2(\xi) \exp(-\xi^2) \ln\left[C_n^2 H_n^2(\xi) \exp(-\xi^2)\right] d\xi + \ln\hbar . \tag{14}$$

Note that for the vacuum energy with $n = 0$, and the first excited state $n = 1$, the values of quantum position and momentum entropies can be determined exactly yielding (see also [18])

$$S_q^{(x)}(0) = S_q^{(p)}(0) = \tfrac{1}{2}(1 + \ln \pi) \tag{15}$$

and

$$S_q^{(x)}(1) = S_q^{(p)}(1) = \tfrac{1}{2} \ln \pi + \ln 2 + \gamma - \tfrac{1}{2} \tag{16}$$

where for the sake of simplicity we have put $m = \omega = \hbar = 1$ and $\gamma \approx 0.5772$ is the Euler constant.

2.2. Position entropy for the classical harmonic oscillator

Suppose that we do not know *a priori* the position of the mass point of a classical harmonic oscillator and we have to determine the probability of finding it in a certain coordinate element $d\xi$. In order to compare the quantum and classical oscillator we suppose that the classical oscillator takes only the energies of the quantum one. To find this probability we have to determine the velocity by means of which we can find the time interval in which the mass element of CHO dwells in a certain coordinate element. For this purpose we use the energy equation of CHO in which we insert quantum energies [11]

$$\dot{x}^2 + \omega^2 x^2 = \frac{\hbar \omega}{m}(2n + 1). \tag{17}$$

From equation (17) one gets for the amplitude of the classical oscillator

$$A_n = \sqrt{\hbar/(m\omega)}\sqrt{2n + 1} \qquad n = 1, 2, \ldots \tag{18}$$

and for the velocity

$$\dot{x} = \omega \sqrt{A_n^2 - x^2}. \tag{19}$$

The time of occurrence dt in the coordinate element dx for the mass point of CHO is given as

$$dt = \frac{dx}{\omega \sqrt{A_n^2 - x^2}}. \tag{20}$$

Since the classical probability dP that the mass point occurs in the coordinate element $d\xi$ is directly proportional to dt, we have

$$\rho_n(x) = \frac{1}{\pi \sqrt{A_n^2 - x^2}}. \tag{21}$$

Inserting of the probability density of CHO we get for the position entropy of CHO

$$S_c^{(x)}(n) = -\int_{-A_n}^{A_n} \rho_n(x) \ln \rho_n(x) \, dx$$

$$= -\frac{1}{\pi A_n} \int_{-A_n}^{A_n} \frac{1}{\sqrt{1 - (x/A_n)^2}} \left[\ln \left(\frac{1}{\pi A_n} \right) + \ln \left(\frac{1}{\sqrt{1 - (x/A_n)^2}} \right) \right] dx. \tag{22}$$

Using the substitution $\zeta = x/A_n$ we have

$$S_c^{(x)}(n) = \tfrac{1}{2} \ln \frac{2\pi^2 \hbar}{m\omega} + C + \tfrac{1}{2} \ln \left(n + \tfrac{1}{2} \right) \tag{23}$$
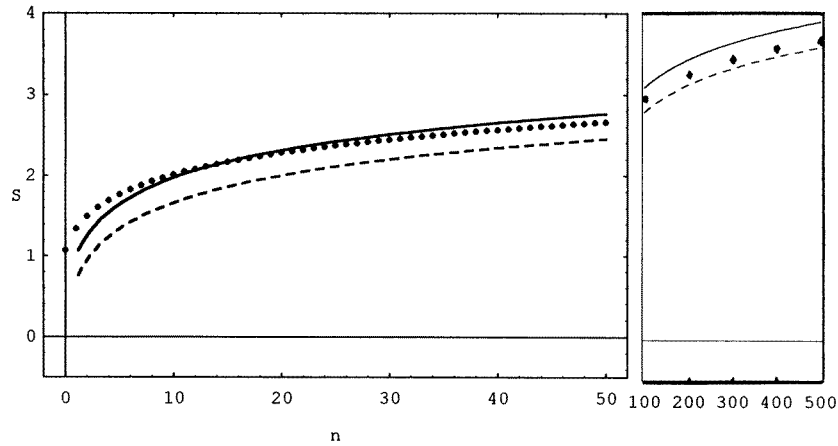
Figure 1. Dependence of the position entropy on the quantum number n for the quantum harmonic oscillator (dotted curve), classical oscillator (full curve) and the asymptotic dependence (31) for the quantum oscillator (broken curve).

where

$$C = \frac{1}{2\pi} \int_{-1}^{1} \frac{\ln(1 - \zeta^2)}{\sqrt{1 - \zeta^2}} d\zeta . \tag{24}$$

The integral (24) can be be calculated by making the change of variable $\zeta = \cos\theta$, which leads to

$$C = \frac{1}{\pi} \int_{0}^{\pi} \ln(\sin(\theta)) d\theta = -\ln 2 . \tag{25}$$

With $C = -\ln 2$, equation (23) get the form

$$S_c^{(x)}(n) = \frac{1}{2} \ln \frac{\pi^2 \hbar (n + 1/2)}{2m\omega} . \tag{26}$$

We have calculated numerically the position entropy of QHO as a function of its energy for the quantum number $n \in [0, 50]$ and for $n = 100, 150, 200, 250$ and 300. (The calculations were performed using *Mathematica* [12]). Numerical values of the entropies are also given in [18, 19]. The calculated values of position entropy are plotted against the quantum number n in figure 1. We see that $S_q^{(x)}(n)$ represents a monotonically increasing function of quantum number n having minimal value $S_q^{(x)}(0)$. Since $S_q^{(x)}(n)$ is equal to $S_q^{(p)}(n)$ the sum $S_q^{(x)}(n) + S_q^{(p)}(n) \equiv S_q^{(x,p)}(n) = 2S^{(x)}(n)$. It is interesting to compare the quantum position entropy $S_q^{(x)}$ with the position entropy of a classical oscillator and find the energy values for which both entropies approach each other. The function $S_q^{(x)}(n) \equiv F(n)$ can be estimated by taking into account the form of the quantum probability densities of QHO. The first estimate of this function can be made if one considers that for high n the position density function consists of a series of peaks having their mean values given by the density function of the classical oscillator. Thus the area where we can find the mass point is approximately one half of the area of the classical oscillator. The information for the position is therefore less by factor of $\ln 2$ than the classical position entropy. A better estimate can be made by also taking into account the shape of the peaks (i.e. not only is the mass point in the peak—yes or no). Then we can write

$$\rho_q(x) \approx 2\cos^2(k_n x)\rho_c(x) \tag{27}$$

where k_n is a high number, increasing with n. The form of the right-hand side of equation (27) is a consequence of the asymptotical formula for Hermite polynomials [21]. Since $\cos(k_n x)$ changes quicker than $\rho_c(x)$, the entropy can be written as

$$S_q^{(x)} \approx - \int \rho_c(x) 2 \cos^2(k_n x) \ln[\rho_c(x) 2 \cos^2(k_n x)] \, \mathrm{d}x \,. \tag{28}$$

Using the identity $2\cos^2 t = 1 + \cos(2t)$, we get

$$S_q^{(x)} \approx - \int \rho_c(x) \ln \rho_c(x) \, \mathrm{d}x - \ln 2 - \int \rho_c(x) \ln \rho_c(x) \cos(2k_n x) \, \mathrm{d}x$$
$$- \ln 2 \int \rho_c(x) \cos(2k_n x) \, \mathrm{d}x - 2 \int \rho_c(x) \cos^2(k_n x) \ln[\cos^2(k_n x)] \, \mathrm{d}x \,. \tag{29}$$

Due to the fast oscillating behaviour of $\cos(2k_n x)$ which takes both signs, the third and fourth terms of (29) are approximately zero. Therefore equation (29) differs from our first estimate by the last integral, which is essentially the mean of the function $\cos^2(k_n x) \ln[\cos^2(k_n x)]$. Since this function oscillates much faster than $\rho_c(x)$ changes, we can approximate the last integral in (29) by averaging this function over its period

$$- 2 \int \rho_c(x) \cos^2(k_n x) \ln[\cos^2(k_n x)] \, \mathrm{d}x \approx -2 \frac{1}{\pi} \int_0^\pi \cos^2 t \ln(\cos^2 t) \, \mathrm{d}t = 2 \ln 2 - 1 \,. \tag{30}$$

Thus

$$S_q^{(x)} \approx \frac{1}{2} \ln\left(\frac{\pi^2 \hbar (2n+1)}{m\omega} \right) - 1 \equiv S_q^{(x)[\mathrm{approx}]} \,. \tag{31}$$

Equation (31) has been previously derived in [17, 19] following a different approach that makes no reference to the classical entropy. The values of $S_c^{(x)}$, $S_q^{(x)}$ and $S_q^{(x)[\mathrm{approx}]}$ as functions of quantum number n are shown in figure 1. We can see that the values of $S_q^{(x)}(n)$ are for $(n \geqslant 15)$ embedded between the values of position entropy of CHO and the values given by the approximate formula (31); $S_q^{(x)}(n)$ approaches the approximate values with increasing n. This points out that a QHO can be treated as a CHO when its quantum number is sufficiently high. The relatively slow convergence to the approximate formula (31) can be explained taking into account the behaviour of the probability densities near the turning points (see figure 2, where $-\rho(\xi) \ln \rho(\xi)$ is depicted). The change of $\rho_c(x)$ becomes not negligible in comparison with the oscillating $\rho_q(x)$ and the quantum density $\rho_q(x)$ has non-zero tails outside the classical interval, which increases the entropy.

The phenomenological function $S_q^{(x)[\mathrm{approx}]}$ also makes it possible to determine the position entropy–energy relation of QHO for higher quantum numbers n. The entropy–energy relation is defined as the ratio $q(n)$ of the entropy change to the energy change between two neighbouring quantum states. Since in the case of QHO there the same energy $\hbar\omega$ is always needed, we can write the asymptotical entropy–energy relation in the form

$$q(n) = k_\mathrm{B} \frac{S_q^{(x)}(n+1) - S_q^{(x)}(n)}{\hbar\omega} \approx \frac{k_\mathrm{B}}{2\hbar\omega} \ln\left(1 + \frac{1}{n+1/2} \right) \approx \frac{k_\mathrm{B}}{2\hbar\omega n} \tag{32}$$

for $n \gg 1$, where k_B is Boltzmann's constant. On the other hand, the maximum change of entropy occurs when the QHO passes from the ground state to the first excited state. This can be calculated exactly yielding

$$q(0) = k_\mathrm{B} \frac{S_q^{(x)}(1) - S_q^{(x)}(0)}{\hbar\omega} = k_\mathrm{B} \frac{\ln 2 + \gamma - 1}{\hbar\omega} \approx k_\mathrm{B} \frac{0.2703}{\hbar\omega} \,. \tag{33}$$
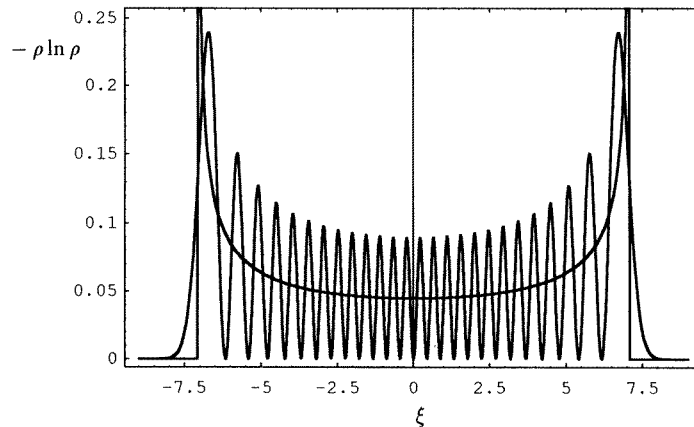
Figure 2. Quantum and classical harmonic oscillator values of $-\rho(\xi) \ln \rho(\xi)$ for the quantum number $n = 25$.

3. Entropy of non-stationary states

Up to now we were dealing with entropy of QHO Hamiltonian eigenstates. It is also interesting to see how the entropy evolves in the case of non-stationary states of the oscillator. The simplest class of such states are the coherent states. They can be defined as a shifted vacuum, and therefore both the position and the momentum probability distributions are simply shifted. Thus the entropies are the same as for the vacuum (i.e. $n = 0$ state):

$$S^{(x)}(t) = S^{(p)}(t) = \tfrac{1}{2}(1 + \ln \pi). \tag{34}$$

(We assume $\hbar = m = \omega = 1$ for simplicity.) The situation is more interesting for other simple states—squeezed vacuum and the so-called Schrödinger cat states.

3.1. Squeezed vacuum

Let us assume that for $t = 0$ the wavefunction of the oscillator is

$$\psi(x, 0) = (s^2 \pi)^{-1/4} \exp\left(-\frac{x^2}{2s^2}\right) \tag{35}$$

where $s \in (0, \infty)$ is the squeezing parameter. As we can see, for $s = 1$ we obtain a vacuum state, while for $s > 1$ the corresponding Gaussian curve is 'stretched' and for $s < 1$ the curve is 'squeezed'. We can expect that the position entropy for $s > 1$ ($t = 0$) will be larger and for $s < 1$ less, respectively, than in the case of a vacuum state. We remark that from the viewpoint of the 'usual' UR equation (34) still describes the minimum uncertainty state.

We are now interested in the time evolution of wavefunction (35). This wavefunction can be found in arbitrary time using the Green function of the LHO:

$$\psi(x, t) = \int G(x, x', t) \psi(x', 0) \, \mathrm{d}x' \tag{36}$$

where [13, 14]

$$G(x, x', t) = \frac{\mathrm{e}^{\mathrm{i}\pi/4} \, \mathrm{e}^{\mathrm{i}t/2}}{\sqrt{2\pi \sin t}} \exp\left(\mathrm{i} \frac{x^2 \cos t - 2xx' + x'^2 \cos t}{2 \sin t}\right). \tag{37}$$

Calculating the integral (36) we get

$$\psi(x, t) = \frac{e^{i(t/2 + \pi/4)}}{\sqrt[4]{\pi}\sqrt{(1/s)\sin t - is\cos t}}\exp\left(-i\frac{(s^2 - 1/s^2)\sin 2t}{4\left((1/s^2)\sin^2 t + s^2\cos^2 t\right)}x^2\right)$$

$$\times \exp\left(-\frac{x^2}{2\left(1/s^2\right)\sin^2 t + s^2\cos^2 t}\right). \tag{38}$$

The probability distribution is then

$$\rho(x, t) = \frac{1}{\sqrt{\pi}\sigma_x(t)}\exp\left(-\frac{x^2}{\sigma_x^2(t)}\right) \tag{39}$$

where

$$\sigma_x(t) = \sqrt{(1/s^2)\sin^2 t + s^2\cos^2 t}\,. \tag{40}$$

Now we can easily calculate the position entropy as a function of time

$$\begin{aligned}S^{(x)}(t) &= \tfrac{1}{2}(1 + \ln\pi) + \ln[\sigma_x(t)]\\ &= \tfrac{1}{2}\left[1 + \ln\pi + \ln\left((1/s^2)\sin^2 t + s^2\cos^2 t\right)\right].\end{aligned} \tag{41}$$

As we can see, for $s < 1$ the position entropy oscillates between its minimum value

$$S_{\min}^{(x)} = \tfrac{1}{2}(1 + \ln\pi) + \ln s \tag{42}$$

for $t = 0,\ \pi,\ 2\pi\ldots$ and its maximum value

$$S_{\max}^{(x)} = \tfrac{1}{2}(1 + \ln\pi) - \ln s \tag{43}$$

for $t = \pi/2,\ 3\pi/2\ldots$. Keeping in mind that for the QHO $\rho(p, t) = \rho(x, t + \pi/2)$, the momentum entropy $S^{(p)}(t)$ can be expressed as

$$\begin{aligned}S^{(p)}(t) &= \tfrac{1}{2}(1 + \ln\pi) + \ln[\sigma_x(t + \pi/2)]\\ &= \tfrac{1}{2}\left[1 + \ln\pi + \ln\left(s^2\sin^2 t + (1/s^2)\cos^2 t\right)\right].\end{aligned} \tag{44}$$

The sum of both entropies is then

$$S^{(x)}(t) + S^{(p)}(t) = 1 + \ln\pi + \frac{1}{2}\ln\left(\frac{3 + a}{4} - \frac{a - 1}{4}\cos(4t)\right) \tag{45}$$

where $a \equiv (s^4 + 1/s^4)/2$. This function is periodic with a period $\pi/2$; its minimum value is

$$S_{\min} = 1 + \ln\pi \tag{46}$$

i.e. equal to the vacuum state value. Its maximum value is

$$S_{\max} = 1 + \ln\pi + \tfrac{1}{2}\ln\left(\frac{a + 1}{2}\right). \tag{47}$$

Thus, even though each of the partial entropies $S^{(x)}$ and $S^{(p)}$ can take arbitrary low values (choosing appropriate s), their sum never drops below the vacuum state value, i.e. the Bia-lynicki-Birula–Mycielski limit (5). The time evolution of the entropies for the squeezed vacuum is shown in figure 3.
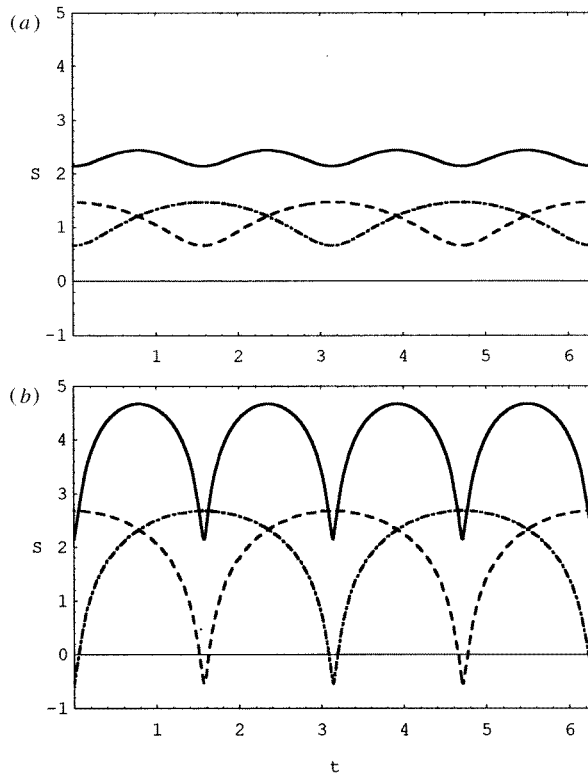
Figure 3. Time evolution of the position entropy (broken curve), momentum entropy (chain curve) and their sum (full curve) for the squeezed vacuum. Parameter s is (a) $s = 1.5$ and (b) $s = 5$.

3.2. Schrödinger cat states

Schrödinger cat states are quantum superpositions of macroscopically distinguishable 'classical' states. In quantum optics these states are usually considered as superpositions of two coherent states with the same amplitude but with opposite phases (see e.g. [15]), i.e.

$$|\psi\rangle = A\left(|\alpha\rangle + \mathrm{e}^{-\mathrm{i}\varphi}|-\alpha\rangle\right).\tag{48}$$

Here $|\pm\alpha\rangle$ are the coherent states which can be written in the coordinate representation in the form

$$\langle x|\pm\alpha(t)\rangle = \pi^{-1/4}\exp\left(-\frac{(x\mp x_0)^2}{2}\pm\mathrm{i}p_0 x\right)\tag{49}$$

where $\pm x_0$ and $\pm p_0$ are the mean positions and momenta of the coherent states evolving in time according to the equations

$$x_0 = a\cos t \qquad p_0 = -a\sin t.\tag{50}$$

The normalization constant is given by the equation

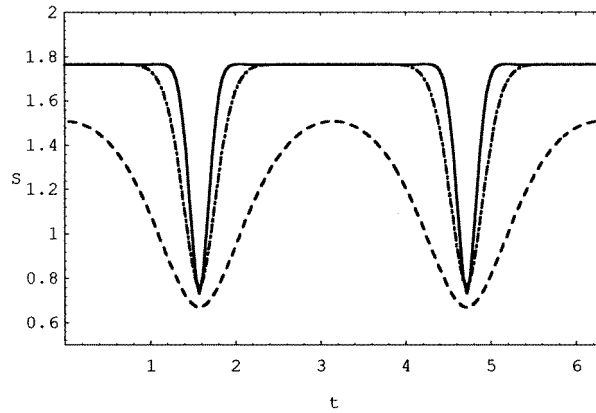$$|A|^2 = \frac{1}{2(1 + \exp(-a^2)\cos\varphi)}\tag{51}$$

Figure 4. Time evolution of the position entropy for the Schrödinger-cat state (48) with $\varphi = 0$ (an even state). The parameter a is $a = 1$ (broken curve), $a = 3$ (chain curve) and $a = 5$ (full curve).

where φ is the relative phase of the superposition. For $\varphi = 0$ the superposition is called an even state (in Fock representation it contains only even numbers of photons) and similarly for $\varphi = \pi$ the state is called an odd state. The probability density is of the form

$$\rho(x, t) = \frac{2|A|^2}{\sqrt{\pi}} \left(e^{-(x^2 + x_0^2)} (\cosh(2x_0 x) + \cos(2p_0 x + \varphi)) \right). \tag{52}$$

The entropy of such states cannot be calculated exactly, but we can make a reasonable estimate for the case when a is sufficiently high. For time t around zero, when the probability distribution essentially consists of two separated Gaussian peaks the entropy is approximated by the term $\ln 2$, larger than for the coherent state value (34), i.e.

$$S_{\max}^{(x)} \approx \tfrac{1}{2}(1 + \ln \pi) + \ln 2 \approx 1.766. \tag{53}$$

We see that we need information $\ln 2$ (in nats, that corresponds to one bit), for determining in which peak the particle occurs. On the other hand, if t approaches $\pi/2$, the peaks overlap and create an interference pattern. For $t = \pi/2$ the probability distribution is a Gaussian modulated highly oscillating squared sinus function. For its entropy we can use the estimate (30), which yields the minimum value

$$S_{\min}^{(x)} \approx \tfrac{1}{2}(1 + \ln \pi) + \ln 2 - 1 \approx 0.766. \tag{54}$$

We see that these extremal values differ from each other by the value 1. Numerical calculations were performed for several values of a (see figure 4), and except at very low values of a (when the Gaussians always overlap), the agreement with our previous estimate was very good.

Similarly the momentum entropy can also be calculated, simply by shifting the time argument by $\pi/2$. From the numerical calculations as well as from the estimates it follows that the sum of position and momentum entropies never drops below the vacuum state value.

Acknowledgments

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